Period 1, Oct 7, 2024

5. The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30-t)^2$.

a) How fast is the water running out at the end of 10 minutes?

$\frac{dQ(T)}{dT} = 200\,(2\,(30-T)' \cdot (-1))$

$200 \cdot [2(30-10)'(-1)]$

$200 \cdot 2 \cdot 20 \cdot -1 = -8000$ g/min

b) What is the average rate at which the water flows out during the first 10 minutes?

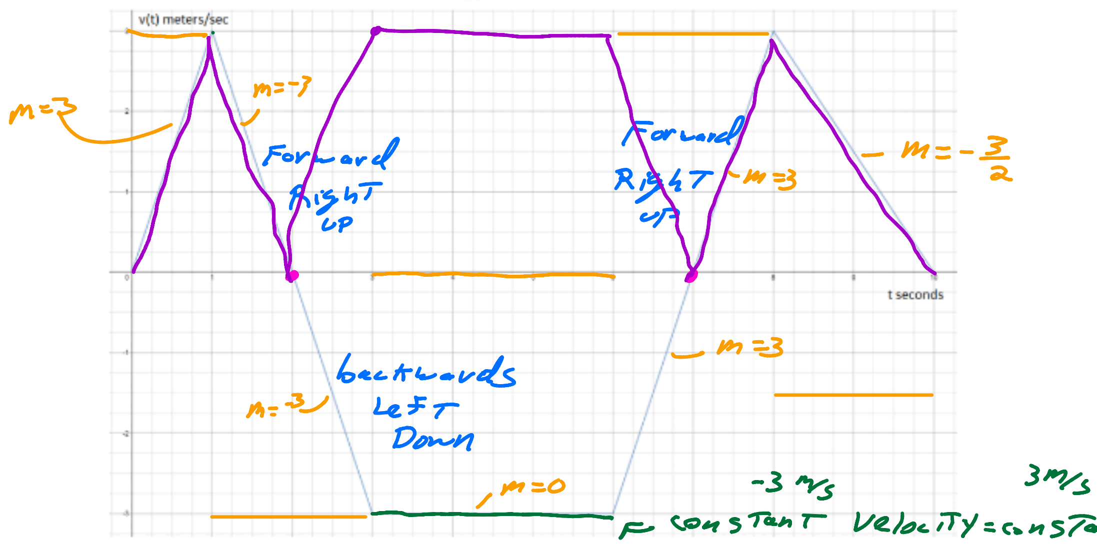
$Q(0) = 200(30-0)^2 = 200 \cdot 900 = 180{,}000$

$Q(10) = 200(30-10)^2 = 200 \cdot 400 = 80{,}000$

$(0, 180{,}000)$ $(10, 8000)$

Slope between 2 pts

2. The accompanying figure shows the velocity $v = \frac{ds}{dt} = f(t)$ meters/sec of a body moving along a coordinate line.



constant velocity = constant speed

a) When does the body reverse direction?

2 sec, 7 sec

Speed = |velocity|

b) When (approximately) is the body moving at a constant speed?

c) Graph the body's speed for $0 \leq t \leq 10$ on the same graph in a different color.

d) Graph the acceleration, where defined, on the same grid in a different color.

Slope of Velocity

a(T)

1. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s(t) = 24t - 0.8t^2$ meters in t seconds.

a) Find the rock's velocity and acceleration as functions of time. (The acceleration in this case is the acceleration of gravity on the moon).

b) How long did it take the rock to reach its highest point?

c) How high did the rock go?

d) When did the rock reach half its maximum height? Max =180 , Half Max=90 < way up / way Down

$S(T)=90=24T-0.8T^2 \Rightarrow 0.8T^2-24T+90=0$

$a=0.8 \quad b=-24 \quad c=90$

$T=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$T=\dfrac{-(-24)\pm\sqrt{(-24)^2-4(0.8)90}}{2(0.8)}$

e) How long was the rock aloft?

Lands $S(T)=0$ > on ground

Takes off $S(T)=0$

$S(T)=0=24T-0.8T^2$

$0=T(24-0.8T)$

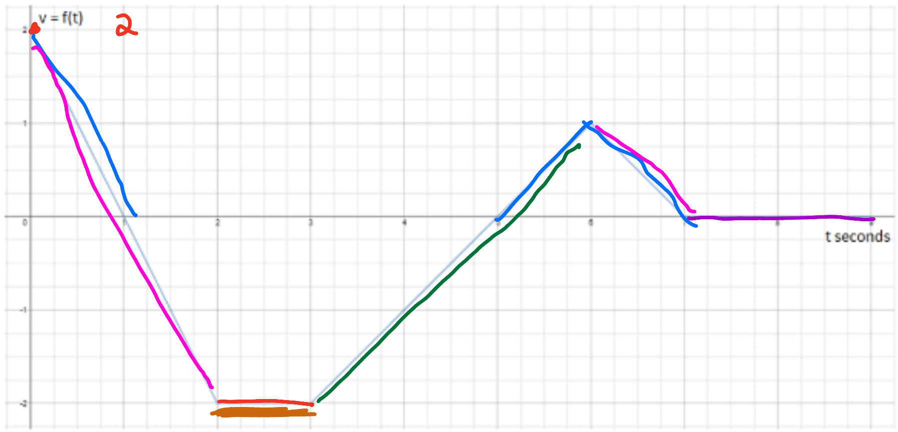
$T=0$ or $24-\cancel{0.8T}=0$

$+\cancel{0.8T} \quad 0.8T$

$\dfrac{24}{0.8}=\dfrac{\cancel{0}.8T}{0.\cancel{8}}$

$30\text{sec}=T$

4. The accompanying figure shows the velocity $v = \frac{ds}{dt} = f(t)$ meters/sec of a body moving along a coordinate line.



a) When does the particle move forward? Move backward? Speed up? Slow down?

$(0,1) \cup (5,7)$ $(1,5)$

b) When is the particle's acceleration positive? Negative? Zero?

$(3,6)$ $(0,2) \cup (6,7)$

Slope of velocity $(2,3)$

c) When does the particle move at its greatest speed? Speed = |velocity|

$V(0) = \text{Speed}(0) = 2\ m/s$

$|V(2 \text{ to } 3)| = \text{Speed}(2 \text{ to } 3) = 2\ m/s$

$|-2| = 2$

d) When does the particle stand still for more than an instant?

velocity = 0

$(7,9)$

$\left(\frac{1}{u}\right)^2 = \frac{1}{u^2} = u^{-2}$

$-2-1 = -3$

19. $f(t) = \left(\frac{1}{t-3}\right)^2$

$u = T-3$

$\frac{du}{dT} = 1$

$y = \left(\frac{1}{u}\right)^2 = u^{-2}$

$\frac{dy}{du} = -2u^{-3}$

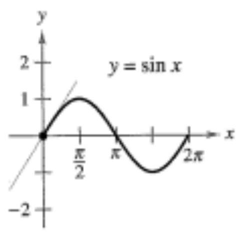
$\frac{\cancel{du}}{dT} \cdot \frac{dy}{\cancel{du}}$

$1 \cdot -2u^{-3} = \frac{-2}{u^3} = \frac{-2}{(T-3)^3}$

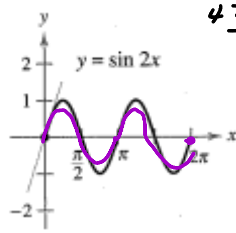
In Exercises 43 and 44, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

$\cos 0 = 1$

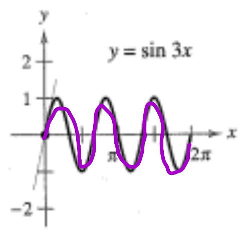
43. (a)



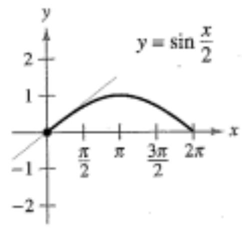
(b)



44. (a)



(b)



43 a) $\frac{dy}{dx} = \cos x = \underline{1}$

b) $\frac{dy}{dx} = 2\cos 2x = 2 \cdot 1 = 2$ periods

44 a) $\frac{dy}{dx} = 3\cos 3x = 3 \cdot 1 = 3 = 3$ periods

b) $\frac{dy}{dx} = \frac{1}{2}\cos\frac{x}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\frac{1}{2}$ period

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35. $f(x) = \sqrt{2 + \sqrt{2 + \sqrt{x}}}$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$y = \sqrt{2 + \sqrt{2+u}}$

$L = \sqrt{2+u}$

$\frac{dL}{du} = \frac{1}{2\sqrt{2+u}}$

$\Rightarrow y = \sqrt{2+L}$

$\frac{dy}{dL} = \frac{1}{2\sqrt{2+L}}$

$\frac{\cancel{du}}{dx} \cdot \frac{\cancel{dL}}{\cancel{du}} \cdot \frac{dy}{\cancel{dL}}$

$\frac{1}{2\sqrt{x}} \cdot \frac{1}{2\sqrt{2+u}} \cdot \frac{1}{2\sqrt{2+L}} = \frac{1}{2\sqrt{x}} \cdot \frac{1}{2\sqrt{2+\sqrt{x}}} \cdot \frac{1}{2\sqrt{2+\sqrt{2+\sqrt{x}}}}$

33. $f(x) = ((x^2+3)^5 + x)^2 \Rightarrow$ $F'(x) = 2((x^2+3)^5+x)(5(x^2+3)^4 \cdot 2x+1)$

$u = x^2+3$ $\quad$ $y = (x^2+3)^5$ $\quad$ $L = (x^2+3)^5 + x$ $\quad$ $y = (L)^2 = L^2$

$\frac{du}{dx} = 2x$ $\quad$ $y = u^5$ $\quad$ $\frac{dL}{dx} = 10x(x^2+3)^4+1$ $\quad$ $\frac{dy}{dL} = 2L$

$\frac{dy}{du} = 5u^4$

$\frac{\cancel{du}}{dx} \cdot \frac{dy}{\cancel{du}} = 2x \cdot 5u^4 = 10x(x^2+3)^4$

$\frac{\cancel{dL}}{dx} \cdot \frac{dy}{\cancel{dL}} = [10x(x^2+3)^4+1][2((x^2+3)^5+x)]$

$\int \sin^2 x \, dx$

$h(x) = F(x) + g(x) \Rightarrow h'(x) = F'(x) + g'(x)$

64. $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

$y = \sin \sqrt[3]{x}$ $\quad \Rightarrow y = \sin u$

$u = \sqrt[3]{x} = x^{\frac{1}{3}}$ $\quad$ $\frac{dy}{du} = \cos u$

$\frac{du}{dx} = \frac{1}{3\sqrt[3]{x^2}}$ $\quad$ $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}} \cdot \cos \sqrt[3]{x}$

$y = \sqrt[3]{\sin x} \Rightarrow y = \sqrt[3]{u} = u^{\frac{1}{3}}$

$u = \sin x$ $\quad$ $\frac{dy}{du} = \frac{1}{3u^{\frac{2}{3}}}$

$\frac{du}{dx} = \cos x$

$\frac{\cos x}{3\sqrt[3]{\sin^2 x}}$

$$\frac{dy}{dx} = \frac{\cos \sqrt[3]{x}}{3\sqrt[3]{x^2}} + \frac{\cos x}{3\sqrt[3]{\sin^2 x}}$$

50. $y = \cos(1-2x)^2$

49. $y = \sin(\pi x)^2 = \sin \pi^2 x^2$

$\frac{dy}{dx} = \cos(\pi x)^2 \cdot 2x\pi^2$

53. $f(x) = \frac{\cot x}{\sin x}$ $\Rightarrow$ $F'(x) = \frac{-\csc^2 x(\sin x) - \cot x(\cos x)}{\sin^2 x}$

$g(x) = \sin^2 x$

$g'(x) = 2\sin x \cos x$

$= \sin 2x$

$F(x) = \frac{\frac{\cos x}{\sin x}}{\frac{\sin x}{1}} = \frac{\cos x}{\sin^2 x} \Rightarrow F'(x) = \frac{-\sin x(\sin^2 x) - (\cos x)(2\sin x \cos x)}{(\sin^2 x)^2}$

$\frac{-\sin x(\sin^2 x + 2\cos^2 x)}{\sin^{3} x}$

$\frac{-(\overbrace{\sin^2 x + \cos^2 x}^{1} + \cos^2 x)}{\sin^3 x}$

61. $f(t) = 3\sec^2(\pi t - 1)$

$y = 3\sec^2(\pi T - 1)$

$u = \pi T - 1$

$\frac{du}{dT} = \pi$

$y = 3\sec^2 u \Rightarrow y = 3L^2$

$L = \sec u$ $\quad \frac{dy}{dL} = 6L$

$\frac{dL}{du} = \sec u \tan u$

$\pi \cdot \sec u \tan u \cdot 6L$

$\pi \cdot \sec(\pi T - 1)\tan(\pi T - 1) \cdot 6\sec(\pi T - 1)$

62. $y = 3x - 5\cos(\pi x)^2$

$y = F(x) - g(x)$

$F(x) = 3x$

$F'(x) = 3$

$\frac{dy}{dx} = 3 - (-10\pi^2 x \sin(\pi x)^2)$

$y = 5\cos(\pi x)^2$

$y = 5\cos \pi^2 x^2$

$u = \pi^2 x^2$

$\frac{du}{dx} = 2\pi^2 \cdot x$

$y = 5\cos u$

$\frac{dy}{du} = -5\sin u$

$2\pi^2 \cdot x \cdot -5\sin(\pi x)^2$

CAS In Exercises 37–42, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

41. $y = \frac{\cos \pi x + 1}{x}$

42. $y = x^2 \tan \frac{1}{x}$

$\frac{dy}{dx} = 2x + \frac{-\sec^2(\frac{1}{x})}{x^2}$

$\frac{dy}{dx} = 2x \cdot \tan\frac{1}{x} + x^2 \cdot \frac{d}{dx}(\tan \frac{1}{x})$

$u = \frac{1}{x} = x^{-1}$ $\quad y = \tan u$

$\frac{du}{dx} = \frac{-1}{x^2}$ $\quad \frac{dy}{du} = \sec^2 u$

58. $g(\theta) = \cos^2 8\theta$ $\Rightarrow y = \cos^2 u$ $\Rightarrow y = L^2$ $\quad 8 \cdot -\sin 8\theta \cdot 2 \cdot \cos 8\theta$

$u = 8\theta$ $\quad L = \cos u$

$\frac{du}{d\theta} = 8$ $\quad \frac{dL}{du} = -\sin u$ $\quad \frac{dy}{dL} = 2L$

66. $y = \cos\sqrt{\sin(\tan \pi x)}$

$y = \cos\sqrt{\sin(\tan \pi x)} \Rightarrow y = \cos\sqrt{\sin(\tan u)} \Rightarrow y = \cos\sqrt{\sin L} \Rightarrow y = \cos\sqrt{m}$

$u = \pi x$ $\quad \frac{du}{dx} = \pi$

$L = \tan u$ $\quad \frac{dL}{du} = \sec^2 u$

$m = \sin L$ $\quad \frac{dm}{dL} = \cos L$

$p = \sqrt{m} = m^{\frac{1}{2}}$ $\quad \frac{dp}{dm} = \frac{1}{2m^{\frac{1}{2}}}$

$y = \cos p$ $\quad \frac{dy}{dp} = -\sin p$

$$\frac{dy}{dx} = \pi \cdot \sec^2 u \cdot \cos L \cdot \frac{1}{2\sqrt{m}} \cdot -\sin p$$

$$\frac{dy}{dx} = \pi \sec^2 \pi x \cdot \cos(\tan \pi x) \cdot \frac{1}{2\sqrt{\sin(\tan \pi x)}} \cdot -\sin\sqrt{\sin(\tan(\pi x))}$$

21. $y = \frac{1}{\sqrt{x+2}}$ $\Rightarrow y = u^{-1}$

$u = \sqrt{x+2}$ $\quad \frac{dy}{du} = \frac{-1}{u^2}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x+2}}$

$$\frac{1}{2\sqrt{x+2}} \cdot \frac{-1}{(\sqrt{x+2})^2}$$

59. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

$u = 2\theta$

$\frac{du}{d\theta} = 2$

$y = \frac{1}{4}\sin^2 u$

$\sin u = L$

$\frac{dL}{du} = \cos u$

$y = \frac{1}{4}L^2$

$\frac{dy}{dy} = \frac{1}{2}L$

27. $y = \frac{x}{\sqrt{x^2+1}}$

$u = x^2+1$ $\quad \frac{du}{dx} = 2x$

$y = \sqrt{x^2+1}$

$y = \sqrt{u} = u^{\frac{1}{2}}$

$\frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{x^2+1}}$

$y = \frac{x}{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = \frac{1 \cdot \sqrt{x^2+1} - x\left(\frac{2x}{2\sqrt{x^2+1}}\right)}{(\sqrt{x^2+1})^2}$$

29. $g(x) = \left(\frac{x+5}{x^2+2}\right)^2$ $\Rightarrow u = \frac{x+5}{x^2+2} \Rightarrow \frac{du}{dx} = \frac{1(x^2+2) - (x+5)(2x)}{(x^2+2)^2}$

31. $f(v) = \left(\frac{1-2v}{1+v}\right)^3$

$$g'(x) = 2\left(\frac{x+5}{x^2+2}\right)\left(\frac{1(x^2+2) - (x+5)(2x)}{(x^2+2)^2}\right)$$

$$F'(v) = 3\left(\frac{1-2v}{1+v}\right)^2\left(\frac{-2(1+v) - (1-2v)(1)}{(1+v)^2}\right)$$

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23. $f(x) = x^2(x-2)^4$

25. $y = x\sqrt{1-x^2}$

$$y = x^2(x-2)^4 \Rightarrow \frac{dy}{dx} = 2x(x-2)^4 + x^2 \cdot 4(x-2)^3 \cdot 1$$

$$y = x(1-x^2)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = 1(1-x^2)^{\frac{1}{2}} + x \cdot \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$$

$$x^2 y + 4y^2 - 3x^5 = 5\sin y$$

with Respect to x $\left(y \Rightarrow \frac{dy}{dx}\right)$

$y = x$

$\frac{dy}{dx} = \frac{dx}{dx}$

$\frac{dy}{dx} = 1$

$$2x \cdot y + x^2 \cdot \frac{dy}{dx} + 4 \cdot 2y^{2-1} \cdot \frac{dy}{dx} - 15x^4 = 5\cos y \cdot \frac{dy}{dx}$$

$$2xy - 15x^4 = 5\cos y\frac{dy}{dx} - x^2\frac{dy}{dx} - 8y\frac{dy}{dx}$$

$$\frac{2xy-15x^4}{5\cos y-x^2-8y}=\frac{\frac{dy}{dx}\cancel{(5\cos y-x^2-8y)}}{\cancel{5\cos y-x^2-8y}}$$

$$\frac{2xy-15x^4}{5\cos y-x^2-8y}=\frac{dy}{dx}$$